Exercise 3

Find $F'(x)$ for the following integrals:

$$F(x) = \int_0^x \sin(x^2 + t^2)\, dt$$

Solution

The Leibnitz rule states that if

$$F(x) = \int_{g(x)}^{h(x)} f(x,t)\, dt,$$

then

$$F'(x) = f(x, h(x))\frac{dh}{dx} - f(x, g(x))\frac{dg}{dx} + \int_{g(x)}^{h(x)} \frac{\partial f}{\partial t}\, dt,$$

provided that f and $\partial f/\partial t$ are continuous. In this exercise, $g(x) = 0$, $h(x) = x$, and $f(x,t) = \sin(x^2 + t^2)$. Applying the rule gives us

$$F'(x) = \sin(2x^2) \cdot 1 - \sin x^2 \cdot 0 + \int_0^x \frac{\partial}{\partial x} \sin(x^2 + t^2)\, dt.$$

Therefore,

$$F'(x) = \sin 2x^2 + 2x \int_0^x \cos(x^2 + t^2)\, dt.$$